Fitzwilliam Maths Circle

Topic: Voting Theory

15th September, 2026

How can a group turn different preferences into a single choice? Write $A \succ B \succ C$ when a voter prefers A to B, and B to C. Every voter gives a complete ranking, with no ties. Only arithmetic and reasoning are needed.

Exercise 1. *First choices*

Thirteen voters choose between A, B and C. Use this table for Exercises 1–3:

Number of voters	4	3	6
Ranking	$A \succ B \succ C$	$B \succ A \succ C$	$C \succ B \succ A$

(a) Under *plurality*, the candidate with the most first-choice votes wins. Who wins? Do they have a *majority*, meaning more than half of all votes?

(b) Suppose B withdraws. Delete B from every ranking and count first choices again. Who wins now, even though nobody changed their mind about A versus C?

Exercise 2. *Same preferences, different rules*

Return to the original election with all three candidates.

(a) The *Borda count* awards 2 points for first place, 1 for second and 0 for third on each ballot. Find every candidate's total. Who has the highest score?

(b) Under *instant runoff*, if nobody has a majority of first choices, eliminate the candidate with the fewest and transfer each of their votes to its next remaining choice. Who wins? Compare the winners under all three rules.

Exercise 3. *Winning every head-to-head contest*

In a head-to-head contest, each voter chooses whichever of the two candidates they rank higher. A *Condorcet winner* beats every other candidate this way.

(a) Find the vote totals for A against B, B against C, and C against A. Is there a Condorcet winner? Did all the rules above elect that candidate?

(b) Prove that a candidate ranked first by more than half the voters must be a Condorcet winner. Must a Condorcet winner be ranked first by more than half?

Exercise 4. *When the majority goes in circles*

Now consider just three voters, with rankings

$$A \succ B \succ C, \qquad B \succ C \succ A, \qquad C \succ A \succ B.$$

(a) Find all three head-to-head winners. Draw an arrow from each winner to the candidate they beat. Can one ranking of A, B, C agree with all three results?

(b) First hold a vote between two candidates, then pit its winner against the third. Everyone votes for the candidate they prefer in each pair. Show that *any* candidate can win overall if you choose the first pair suitably.

Exercise 5. *A small impossibility theorem*

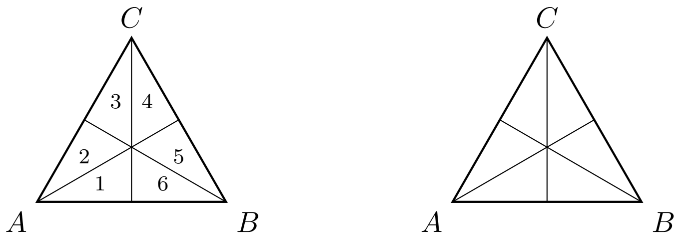
For an odd number of voters, we want a rule that always selects exactly one winner and meets both demands: *with two candidates, the majority's choice wins*; and *removing a losing candidate from every ranking never changes the winner.*

(a) Suppose the rule chooses A in Exercise 4. Which losing candidate could withdraw to force the rule to choose someone else?

(b) Repeat for possible winners B and C. Deduce that no rule can meet both demands for every election, however it breaks ties.

Voting-rule definitions: University of Hawaiʻi, *The Mathematics of Voting.*

The representation triangle. Put A, B and C at the vertices of an equilateral triangle and rank each interior point by proximity: the nearer a vertex, the higher that candidate. The perpendicular bisector of a side is the median from the opposite vertex, so the three medians cut the triangle into six regions, one per strict ranking; a point *on* a median is tied, so the median from C is where $A \sim B$. Region 1 is nearest A and next nearest B, so it holds the $A \succ B \succ C$ voters. Record a profile by writing in each region how many voters hold that ranking.



Exercise 6. *Reading tallies off the triangle*

(a) Name the ranking held in each of regions 2–6, then copy the thirteen voters of Exercise 1 into the blank triangle, shading the three regions they occupy.

(b) Every voter on one side of the median from C ranks A above B, and everyone on the other side ranks B above A. Explain why, then recover the three head-to-head tallies of Exercise 3 by adding the numbers either side of each median. Two regions meet at each vertex and share a first choice; adding each such pair recovers the plurality tally.

Exercise 7. *One rule, one parameter*

A *positional rule* w_s awards 1 point to a voter's first choice, s to their second and 0 to their third, for fixed $s \in [0, 1]$: plurality is w_0, "vote for two" is w_1, and Borda is $w_{1/2}$ once $(2, 1, 0)$ is halved.

(a) Each candidate is ranked second in exactly two regions. Mark them, and show that the w_s tallies for the Exercise 1 profile are $4 + 3s$ for A, $3 + 10s$ for B, and 6 for C.

(b) Confirm that $s = 0$ gives the plurality tally of Exercise 1(a), and that $s = \frac{1}{2}$ gives exactly half the Borda totals of Exercise 2(a), and so the same ranking.

Exercise 8. *Sweeping s from 0 to 1*

(a) Solve for the three values of s at which two tallies are equal, and deduce that this one profile yields four different strict rankings as s runs over $[0, 1]$.

(b) Show that one candidate finishes first under no positional rule at all, yet wins in Exercise 2(b). Which of the rules you have met, then, are *not* positional rules? A theorem of Saari, quoted without proof, bounds the strict positional rankings of an n-candidate profile by $(n - 1)\,[(n - 1)!]$; evaluate that for $n = 8$.

Exercise 9. *Two ways to disturb an election*

(a) Add to the Exercise 1 profile the three voters of Exercise 4, one in each of regions 1, 5 and 3, sitting 120° apart. Each gains one first and one second place, so every tally rises by $1 + s$ and no positional ranking moves, whatever s. But the head-to-head votes *do* move: A gains a net vote on B, B one on C, C one on A. How many copies force all three results to cycle?

(b) Add instead one voter in region 1 and one in region 4, holding the reversed rankings $A \succ B \succ C$ and $C \succ B \succ A$. Show that every head-to-head contest gains a vote on each side, so no margin changes, while the A, B and C tallies rise by 1, $2s$ and 1.

(c) Such a pair leaves every positional ranking alone, in any profile whatever, only if it raises all three tallies equally. Solve $2s = 1$ and name the rule you have found.

Discussion. Plurality elects C in Exercise 1, yet the three B-first voters can all vote A and do better. Where else can voters gain by ranking insincerely?