Fitzwilliam Maths Circle

Topic: Mathematics for AI

2026-07-06

Exercise 1. A population is screened for a disease. Suppose the prior probability that a randomly chosen person has the disease is $p(C = 1) = 0.001$. The screening test is positive for 90% of people who have the disease, and is (falsely) positive for 3% of people who do not. Using Bayes' theorem, compute the probability $p(C = 1 \mid T = 1)$ that a person who tests positive actually has the disease. Comment on why the answer is so small despite the test's apparently high accuracy.

Exercise 2. Consider four six-sided dice with the following faces:

$$A = \{0, 0, 4, 4, 4, 4\}, \quad B = \{3, 3, 3, 3, 3, 3\}, \quad C = \{2, 2, 2, 2, 6, 6\}, \quad D = \{1, 1, 1, 5, 5, 5\}.$$

Two dice are compared by rolling both and seeing which shows the higher number. Arranged in the cyclic order $A \to B \to C \to D \to A$, show that each die beats the next one in the cycle — and that D beats A — with probability $2/3$ in every case. (Such dice are called *non-transitive*, or *Efron dice*: there is no "best" die, since each is beaten by another.)

Exercise 3. The uniform distribution over the interval (c, d) is

$$p(x) = \frac{1}{d - c}, \quad x \in (c, d),$$

and zero elsewhere. Verify that it is correctly normalised, and find expressions for its mean and its variance.

Exercise 4. Verify that each of the following densities is correctly normalised, i.e. that each integrates to 1:

(a) the exponential distribution $p(x \mid \lambda) = \lambda\, e^{-\lambda x}$ for $x \geq 0$;

(b) the Laplace distribution $p(x \mid \mu, \gamma) = \dfrac{1}{2\gamma} \exp\left(-\dfrac{|x - \mu|}{\gamma}\right)$ for $x \in \mathbb{R}$.

Exercise 5. Show that the Gaussian distribution

$$\mathcal{N}(x \mid \mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$

attains its maximum (its *mode*) at $x = \mu$.

Exercise 6. Consider the integral

$$I = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}\,x^2\right) dx.$$

Evaluate it by the following route. First write I^2 as a double integral over x and y. Transform from Cartesian coordinates (x, y) to polar coordinates (r, θ), and then substitute $u = r^2$. Performing the integrals over θ and u, and taking the square root, show that

$$I = \left(2\pi\sigma^2\right)^{1/2}.$$

Hence show that the Gaussian $\mathcal{N}(x \mid \mu, \sigma^2)$ is correctly normalised.

Exercise 7. A function f is said to be *strictly convex* on an interval if every chord joining two points of its graph lies strictly above the graph between those points. Show that this is equivalent to the condition $f''(x) > 0$ throughout the interval.

Exercise 8. The two-point definition of convexity states that, for a convex function f and any $\lambda \in [0, 1]$,

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b).$$

Using proof by induction, show that this implies *Jensen's inequality*: for any points $x_1, \ldots, x_K$ and weights $\lambda_i \geq 0$ with $\sum_i \lambda_i = 1$,

$$f\left(\sum_{i=1}^{K} \lambda_i x_i\right) \leq \sum_{i=1}^{K} \lambda_i f(x_i).$$

Exercise 9. Let $h(p)$ denote the information gained on observing a random variable taking a particular value of probability p. For two independent events of probabilities p and q, information is additive: $h(pq) = h(p) + h(q)$.

(a) Show that $h(p^2) = 2h(p)$, and hence by induction that $h(p^n) = n\,h(p)$ for every positive integer n.

(b) Show that $h\left(p^{n/m}\right) = (n/m)\,h(p)$ for positive integers m, n, so that $h(p^x) = x\,h(p)$ for every positive rational x, and hence, by continuity, for every positive real x.

(c) Conclude that $h(p) \propto \ln p$.

Exercise 10. By applying Jensen's inequality (Exercise 8) with the concave function $f(x) = \ln x$, show that for any positive real numbers $x_1, \ldots, x_n$ the arithmetic mean is never less than the geometric mean:

$$\frac{1}{n}\sum_{i=1}^{n} x_i \;\geq\; \left(\prod_{i=1}^{n} x_i\right)^{1/n}.$$

Sources: Problems adapted from *Deep Learning: Foundations and Concepts* by Christopher M. Bishop and Hugh Bishop (Ch. 2).