Packing Problems

August 6th, 2026

Introduction

In this problem set, we present some (relatively) easier questions related to packing. The set builds up to a central problem - previously open - on packing squares in circles. It was placed on the International Mathematical Summer Camp competition in 2026, and was solved by only 4 of 278 students. The Irish IMO squad solved the problem following the contest.

The difficulty here is (very) not linear: we would encourage people to try all the problems even if they get stuck on one. To get the most out of it, try everything except 2.6 and 3.2 - we'll cover those.

1 Packing Circles

Problem 1.1. Find a good way to pack the Cartesian plane with unit circles, so that as much area as possible is covered. (No proof required.)

Problem 1.2. Think of a good way to pack spheres in infinite 3-dimensional space. Try using some balls to get physical intuition!

Problem 1.3. Take 7 circles of radius 1. What is the minimum value r such that these circles can be packed into a circle of radius r? Hint: try looking at the centres of the circles: can we divide the circle into areas such that each area contains at most one centre?

2 Packing Unit Squares in a Square

Let $s(n)$ denote the minimum side length of a square in which n unit squares can be packed.

Problem 2.1. Find $s(n^2)$.

Try to prove the following technical lemmas before moving on to the problems: we will put them to good use.

Lemma 2.2. A unit square lying in the first quadrant with its center in $[0,t]^2$ must contain the point (t,t).

Lemma 2.3. If a unit square has its center below the line $y=1$ and lies entirely above the x-axis, then the intersection of the line $y=1$ with the square has length at least $2\sqrt{2}-2$.

Lemma 2.4. If the center of a unit square u lies inside a triangle ABC each of whose sides has length at most 1, then u contains A, B, or C.

Now we move on to the meat of this section. The ideas contained here will be very useful for the final section: if you didn't manage the lemmas above, don't worry - we can apply them without proof.

Problem 2.5. Prove that $s(2) = s(3) = 2$.

We split this into two main steps below.

(a) Find a construction showing that the minimum side length is at most 2.

(b) Using Lemma 2.2, find a point that must be contained in *any* unit square placed inside a square of side length less than 2.

Problem 2.6. (Harder) Prove that $s(5) = 2 + \frac{1}{\sqrt{2}}$.

(a) Find a construction achieving the value above. (Hint: the squares are no longer all parallel to the axes.)

(b) Find a set of 4 points within the large square such that every unit square placed inside must cover at least one of them. This can be partially motivated by the previous optimal construction.

(c) Using Lemmas 2.2, 2.3 and 2.4, perform casework on the position of the square's center to finish the problem.

3 Packing Squares in a Unit Circle

Let $c(n)$ denote the maximum side length of a square such that n non-overlapping, equal copies of it can be packed into the unit circle.

Problem 3.1. Find constructions for $c(1) = \sqrt{2}$ and $c(2) = \frac{2}{\sqrt{5}}$.

Problem 3.2. Show these constructions are optimal.

Problem 3.3. (IMSC 2026) Find $c(4)$, with proof.

We'll go over this problem - it is likely the hardest one on this list. The construction is the combination of four squares into a bigger square. Here's a hint to start: for packing squares in squares above, the main idea was to find a set of points such that every square must cover at least one of them. Can we find a similar structure to exploit here?

4 Open Problems

Some closely related problems that are still open!

- **Sphere packing in other dimensions.** What is the densest packing of unit spheres in dimensions other than 1,2,3,8, and 24?
- **The sausage conjecture.** In dimension $d \geq 5$, is the packing with convex hull occupying the smallest volume of n unit balls always a "sausage" (balls arranged in a line)? This is known to be true for $d \geq 42$, but remains open for $5 \leq d \leq 41$.
- **$s(n)$ and $c(n)$ for general n.** Exact values of $s(n)$ and $c(n)$ from Sections 2 and 3 are known only for small n: already $s(10)$ and $c(5)$ are open.