Fitzwilliam Maths Circle

Topic: Sequences, Series, and Power Series

2026-06-08

Exercise 1. Determine whether each sequence converges or diverges. If it converges, find the limit.

(a) $a_n = \sqrt{\dfrac{1+4n^2}{1+n^2}}$

(b) $a_n = \ln(2n^2+1) - \ln(n^2+1)$

(c) $a_n = \left(1 + \dfrac{2}{n}\right)^n$

Exercise 2. Determine whether each series is convergent or divergent. If it converges, find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(e^{1/n} - e^{1/(n+1)}\right)$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

(d) Express $2.\overline{516} = 2.516516516\ldots$ as a ratio of integers.

Exercise 3. For each series, compute a partial sum giving the value to the stated accuracy, and justify your error bound.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$, correct to four decimal places.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$, correct to three decimal places. (Use the Integral Test remainder bound.)

(c) How many terms of $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ would you need to estimate the sum to within 0.01?

Exercise 4. Use the following steps to prove that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$. Let h_n and s_n denote the nth partial sums of the harmonic and alternating harmonic series, respectively.

(a) Show that $s_{2n} = h_{2n} - h_n$.

(b) Take as given that $h_n - \ln n \to \gamma$ as $n \to \infty$ (Euler's constant). Use this with part (a) to deduce $s_{2n} \to \ln 2$.

(c) Conclude that $s_n \to \ln 2$.

Exercise 5. Find the radius of convergence and interval of convergence of each power series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n\,3^n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2\, x^n}{2 \cdot 4 \cdot 6 \cdots (2n)}$

Exercise 6. Find the Maclaurin series for each function and state its radius of convergence.

(a) $f(x) = x\cos\left(\frac{1}{2}x^2\right)$

(b) $f(x) = \ln(4-x)$

(c) $f(x) = \dfrac{x^2}{1+x}$

Exercise 7. Use a power series to approximate $\displaystyle\int_0^{0.2} x\ln(1+x^2)\,dx$ correct to six decimal places. Justify the error bound (e.g. via the Alternating Series Estimation Theorem).

Exercise 8. Show that the function $f(x) = \sum_{n=0}^{\infty} x^n/n!$ satisfies $f'(x) = f(x)$. Deduce that $f(x) = e^x$.

Exercise 9. Starting from $\dfrac{1}{1+t^2} = \displaystyle\sum_{k=0}^{\infty}(-1)^k t^{2k}$ for $|t| < 1$:

(a) Integrate term-by-term from 0 to x to show that for $x \in (-1,1)$,

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

(b) Use part (a) to find a series expression for π. (You may assume the series in (a) also holds at $x = 1$; this is Abel's theorem.)

Exercise 10. Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Show by induction that for $x \neq 0$, $f^{(n)}(x) = P_n(1/x)\,e^{-1/x^2}$ for some polynomial P_n.

(b) Show that $f^{(n)}(0) = 0$ for all $n \geq 0$.

(c) Conclude that the Maclaurin series of f converges everywhere, but does *not* equal $f(x)$ for any $x \neq 0$.

Exercise 11*. In this exercise you will prove that e is irrational.

(a) Show that for any $n \geq 1$,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + E_n, \quad \text{where} \quad 0 < E_n < \frac{3}{(n+1)!}.$$

(b) Suppose for contradiction that $e = a/b$ for some $a, b \in \mathbb{N}$. Choose $n \geq b$ and multiply both sides by $n!$. Show that $n!E_n$ must be a positive integer.

(c) Derive a contradiction from the bound in (a).

Exercise 12*. Evaluate $\displaystyle\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$.

Sources: Problems adapted from *Calculus* by James Stewart (9th ed., Ch. 11) and *Real Analysis: A Long-Form Textbook* by Jay Cummings (Ch. 9). Exercises marked $\star$ are bonus.