Fitzwilliam Maths Circle

Topic: Series

May 4, 2026

Exercise 4.3. Prove that if $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges too.

Exercise 4.4.

(a) Give an example of a series with nonnegative terms where $\sum_{k=1}^{\infty} a_k$ diverges, but $\sum_{k=1}^{\infty} a_k^2$ converges.

(b) Prove that if $\sum_{k=1}^{\infty} a_k$ converges where each $a_k > 0$, then $\sum_{k=1}^{\infty} a_k^2$ converges.

(c) Show by example that part (b) is not true if we do not insist that each $a_k > 0$.

Exercise 4.5. Give an example of a series where $\sum_{k=1}^{\infty} a_k$ converges, but $\sum_{k=1}^{\infty} a_{2k}$ diverges.

Exercise 4.6. Let (a_k) be a sequence which converges to 0. Prove that the series $\sum_{k=1}^{\infty}(a_k - a_{k+1})$ converges to a_1.

Exercise 4.7. Give an example of a series $\sum_{k=1}^{\infty} a_k$ where

- $\sum_{k=1}^{\infty} a_k$ converges,
- $\sum_{k=1}^{\infty} a_k^2$ diverges, and
- $\sum_{k=1}^{\infty} a_k^3$ converges.

Exercise 4.8. Find an estimate for $\sum_{k=1}^{\infty}(-1)^k \frac{1}{7k}$ that is accurate to 0.01.

Exercise 4.12.

(a) Prove that if $\sum_{k=1}^{\infty} a_k$ converges absolutely, and (b_k) is a subsequence of (a_k), then $\sum_{k=1}^{\infty} b_k$ also converges absolutely.

(b) Give an example demonstrating that it is necessary to assume that $\sum_{k=1}^{\infty} a_k$ converges absolutely.

Exercise 4.13.

(a) Prove that if (ka_k) converges to a nonzero real number L, then the series $\sum_{k=1}^{\infty} a_k$ diverges. Give an example to show that the converse is false.

(b) Prove that if $(k^2 a_k)$ converges (to any real number), then the series $\sum_{k=1}^{\infty} a_k$ converges. Give an example to show that the converse is false.

Exercise 4.14. Prove that if $a_k > 0$ for all k and $\sum_{k=1}^{\infty} a_k^2$ converges, then $\sum_{k=1}^{\infty} \frac{a_k}{k}$ converges.

Exercise 4.15. Prove the *Cauchy condensation test*. That is, suppose that (a_k) is a decreasing sequence for which $a_k \to 0$. Prove that $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

Exercise 4.19. Give an example of a divergent series $\sum_{k=1}^{\infty} a_k$ for which $\lim_{k\to\infty}(a_{k+1} - a_k) = 0$.

Exercise 4.20.

(a) Give an example of two divergent series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, such that $\sum_{k=1}^{\infty} a_k b_k$ converges.

(b) Give an example of two convergent series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, such that $\sum_{k=1}^{\infty} a_k b_k$ diverges.

(c) Prove that if $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge absolutely, then $\sum_{k=1}^{\infty} a_k b_k$ converges absolutely.

Exercise 4.22. Consider the sum $\sum_{k=1}^{\infty} a_k$, and define s_n to be this series' n^{th} partial sum; that is, $s_n = \sum_{k=1}^{n} a_k$. The series $\sum_{k=1}^{\infty} a_k$ is called *Cesaro summable* if

$$\lim_{n\to\infty} \frac{s_1 + s_2 + \cdots + s_n}{n}$$

converges.

(a) Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then this series is Cesaro summable.

(b) Prove by example that if $\sum_{k=1}^{\infty} a_k$ is Cesaro summable this does *not* imply that $\sum_{k=1}^{\infty} a_k$ converges.

Exercise 4.28. Prove the *summation by parts* formula. That is, prove that if (a_k) and (b_k) are sequences and $s_n = a_1 + a_2 + \cdots + a_n$, then

$$\sum_{k=j+1}^{n} a_k b_k = s_n b_{n+1} - s_j b_{j+1} + \sum_{k=j+1}^{n} s_k (b_k - b_{k+1}).$$

Bonus Exercise 1. Derive the Maclaurin series for each of the following common expressions: (a) e^x (b) $\sin x$ (c) $\cos x$ (d) $\frac{1}{1-x}$

Bonus Exercise 2. Make your answer from bonus exercise 1 more general by extending to the full Taylor series.

Sources: Problems adapted from *Real Analysis: A Long-Form Mathematics Textbook*, chapter 4, by Jay Cummings.