Fitzwilliam Maths Circle

Topic: Integration

2026-04-06

Exercise 1. Evaluate the Riemann sum for $f(x) = x^2 - x$ on $[0, 2]$ with four equal subintervals and right endpoints. Then use the definition of the definite integral (as a limit of Riemann sums) to compute $\int_0^2 (x^2 - x)\, dx$.

Exercise 2. Find the derivative of the function.

(a) $g(x) = \int_1^x \ln(1 + t^2)\, dt$

(b) $y = \int_0^{\tan x} e^{-t^2}\, dt$

(c) $g(x) = \int_{2x}^{3x} \frac{u^2 - 1}{u^2 + 1}\, du$

Exercise 3. Evaluate the following integrals.

(a) $\int_0^1 (x^e + e^x)\, dx$

(b) $\int \frac{(\arctan x)^2}{x^2 + 1}\, dx$

(c) $\int_0^a x\sqrt{a^2 - x^2}\, dx$

(d) $\int_{-2}^2 (x + 3)\sqrt{4 - x^2}\, dx$

Exercise 4. Suppose f is continuous, $f(0) = 0$, $f(1) = 1$, $f'(x) > 0$, and $\int_0^1 f(x)\, dx = \frac{1}{3}$. Find the value of $\int_0^1 f^{-1}(y)\, dy$.

Exercise 5. Prove that $\int_0^b x^2\, dx = \frac{b^3}{3}$ by considering partitions of $[0, b]$ into n equal subintervals, computing the upper and lower sums explicitly, and taking the limit as $n \to \infty$.

Exercise 6. Consider Thomae's function $h : [0,2] \to \mathbb{R}$, defined by

$$h(x) = \begin{cases} 1/n & \text{if } x \neq 0 \text{ and } x = m/n \in \mathbb{Q} \text{ in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x = 0. \end{cases}$$

Prove that h is integrable on $[0,2]$ by completing the following steps.

(a) Prove that $L(h,P) = 0$ for every partition P.

(b) Given $\varepsilon > 0$, determine whether there are finitely or infinitely many x such that $h(x) > \varepsilon/4$. Explain your answer.

(c) Construct a partition P_ε with $U(h, P_\varepsilon) < \varepsilon$, and prove that it works.

Exercise 7. Suppose f is continuous on $[a,b]$, $f(x) \geq 0$ for all x, and $f(x_0) > 0$ for some $x_0 \in [a,b]$. Prove that $\int_a^b f(x)\,dx > 0$.

Exercise 8. Prove that

$$\int_1^a \frac{1}{x}\,dx + \int_1^b \frac{1}{x}\,dx = \int_1^{ab} \frac{1}{x}\,dx.$$

Exercise 9.

(a) Prove that if f is continuous on $[a,b]$, then there exists some $x_0 \in [a,b]$ such that

$$\int_a^b f(x)\,dx = (b-a) \cdot f(x_0).$$

(b) Give an example showing that this need not hold if f is not continuous.

Exercise 10*⋆*. If f is continuous on $[0,\pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^\pi x\,f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx.$$

Then evaluate $\displaystyle\int_0^\pi \frac{x \sin x}{1 + \cos^2 x}\,dx$.

Sources: Problems from *Calculus* by James Stewart (9th ed., Ch. 5) and *Real Analysis: A Long-Form Textbook* by Jay Cummings (Ch. 8). Exercises marked ⋆ are bonus.