Fitzwilliam Maths Circle

Topic: Combinatorics

March 9^{th}, 2026

Exercise 10.20. Read Fibonacci's biography. Describe the problem that he was studying when he independently discovered the Fibonacci sequence. Then, draw a rabbit family tree to explain this problem, and highlight where the Fibonacci numbers are appearing on your tree.

Exercise 10.22. For the following, assume that each domino covers two adjacent squares of the grid.

(a) How many ways there are to put together 18 dominoes into a 2×9 grid?

(b) How many ways there are to put together $2n$ dominoes into a $2 \times n$ grid?

Exercise 10.24. Abraham de Moivre developed generating functions to solve linear recurrence relations by transforming them into algebraic equations that are easier to manipulate. In this exercise, you will learn how to solve a general second-order linear recurrence relation using generating functions. Note that in this chapter we studied the Fibonacci recurrence, which is a specific example of a second-order linear recurrence.

Fix a $c_1, c_2 \in \mathbb{R}$. Consider the second-order linear recurrence relation given by

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

for $n \geq 2$, with initial conditions $a_0 = B$ and $a_1 = C$.

(a) Define the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Write out the first few terms of the series representation of $A(x)$.

(b) Multiply both sides of the recurrence relation by x^n and sum over all $n \geq 2$. By shifting indices, show that this transforms the recurrence relation into an equation involving the generating function $A(x)$.

(c) Plug in the initial conditions and solve for $A(x)$ in terms of a_0 and a_1. Show that $A(x)$ can be written in the form

$$A(x) = \frac{B + (C - c_1 B)x}{1 - c_1 x - c_2 x^2}.$$

(d) Show that this answer coincides with the answer we found for the Fibonacci sequence.

Exercise 10.25. Leonhard Euler famously used generating functions to study integer partitions. Integer partitions are an important problem in combinatorial number theory. Given an integer n, a partition of n is a way to write n as a sum of positive integers where order does not matter. For example, 3 can be partitioned in three ways:

$$3, \qquad 2+1, \qquad 1+1+1.$$

(a) The integer 4 can be partitioned in five ways. List all of them.

(b) The number of partitions of an integer n is denoted $p(n)$. Determine $p(n)$ for $n \in \{1, 2, 3, 4, 5, 6\}$.

(c) It quickly becomes very difficult to determine $p(n)$ exactly. Reflect on why this is the case. Also, look up the value of $p(100)$ and write it down.

(d) Euler was the first to apply generating functions to the study of integer partitions. First, explain why

$$\frac{1}{1-x^k} = 1 + x^k + x^{2k} + x^{3k} + \dots.$$

(e) Defining $p(0)$ to be equal to 1, the generating function for $p(n)$ is

$$p(0) + p(1)x + p(2)x^2 + p(3)x^3 + p(4)x^4 + \dots.$$

Use part (b) to write out this generating function with the explicit coefficients included, up to the x^6 term.

(f) Euler claimed that the partition function's generating function can be written compactly as $\displaystyle\prod_{k=1}^{\infty} \frac{1}{1-x^k}$, which by part (d) is equivalent to

$$(1 + x + x^2 + x^3 + \dots) \cdot (1 + x^2 + x^4 + x^6 + \dots) \cdot (1 + x^3 + x^6 + x^9 + \dots) \cdots.$$

This is called *Euler's formula.* Multiply out this second expression to the point that you can check that its first four terms match your answer in part (e).

(g) Euler then used this formula to discover many partition identities, such as when one insists that the parts in the partition are all odd, or are all distinct, or are no bigger than a number k. He also used it to prove his pentagonal number theorem. Research the math behind one of these and explain how it follows from Euler's formula.

(h) Research the history and the math of how the more modern mathematicians G. H. Hardy and Srinivasa Ramanujan used Euler's product formula as a step towards proving the Hardy–Ramanujan asymptotic formula for $p(n)$.

Exercise 10.26. In Leonhard Euler's 1741 paper on Eulerian paths and tours, he also proves what is today called the *handshaking lemma.* He proves that if G is a graph with e edges, and if V is the set of vertices of G and $\deg(v)$ is the degree of vertex v, then

$$2e = \sum_{v \in V} \deg(v).$$

Prove this lemma.

Exercise 10.27. Is it possible to have a graph with just one vertex of odd degree? Explain why or why not.

Exercise 10.28. The four-color theorem says that every map can be colored with four colors. In an effort to show one difficulty with proving this result, Arthur Cayley constructed a map and colored it with four colors, and then showed that there is a way to add a fifth region to the map which would require a fifth color if you insist on keep the coloring of the original map.

(a) Find your own example of this phenomenon. That is, construct a map, color that map with four colors, and then add a new region to your map and explain why that new region cannot use one of the original four colors.

(b) Take your final map from part (a)—the one with the new region—and find a way to recolor it with four colors.

(c) Explain why this phenomenon suggests a difficulty with attempting to prove the four-color theorem by induction.

Exercise 10.32. Prove that every 2-coloring the edges of K_6 produces a monochromatic K_3. Also, prove that not every 2-coloring of the edges of K_5 produces a monochromatic K_3.

Bonus Exercise 1. Let $w(r, k)$ be the van der Waerden number, i.e. the smallest value of N for which every r-colouring of 1, 2, 3, ... N contains a monochromatic k-term arithmetic progression.

(a) Give some intuition for van der Waerden's proof that *w(r,k)* exists for all r and k

(b) Why is the Ackermann function introduced as a bound on the growth of $w(r, k)$?

(c) Do some research into whether upper bounds on w(r,k), or the function itself, might be μ-recursive but not primitive recursive. How can we nonetheless be sure that w(r,k) is Turing computable?

(d) Why do you think that the gap between the lower and upper bounds on the van der Waerden numbers has remained so large?

Bonus Exercise 2. Let K_n be a complete graph. Let $R(m, n)$ be the Ramsey number for a monochromatic size-n subgraph of K_n.

(a) What is Ramsey's theorem, in terms of $R(m, n)$?

(b) Do some research to understand why it's so extraordinarily difficult to actually calculate any *specific* values of $R(m, n)$.

(c) If 6 people attend Fitzwilliam Maths Circle, it's guaranteed that either 3 of them all know each other, or 3 of them are complete strangers. State this in terms of graph colourings and Ramsey numbers $R(m, n)$.

(d) Why is Ramsey's theorem sometimes summarised as "complete disorder is impossible"?

Sources: Problems adapted from *Math History: A Long-Form Mathematics Textbook* by Jay Cummings, chapter 10.