Fitzwilliam Maths Circle

Topic: Topology

2026-02-07

These exercises explore Euler's polyhedron formula, $V - E + F = 2$, which relates the number of vertices V, edges E, and faces F of any convex polyhedron (or, more generally, any connected planar graph).

Exercise 9.1.

(a) Can a polyhedron have 8 faces, 25 edges, and 16 vertices?

(b) Can a polyhedron have V vertices, E edges, and F faces such that these three numbers form an arithmetic progression?

Exercise 9.2. In this exercise, you will prove that no polyhedron has precisely 7 edges. To begin, suppose you have a polyhedron with V vertices, E edges, and F faces.

(a) By thinking geometrically, prove that $3F \leq 2E$.

(b) By thinking geometrically, prove that $3V \leq 2E$.

(c) Using parts (a) and (b) and the fact that every polyhedron must have more than 3 faces, conclude that $F = 4$ and $V = 4$.

(d) Using a theorem from this chapter, conclude that no polyhedra have precisely 7 edges.

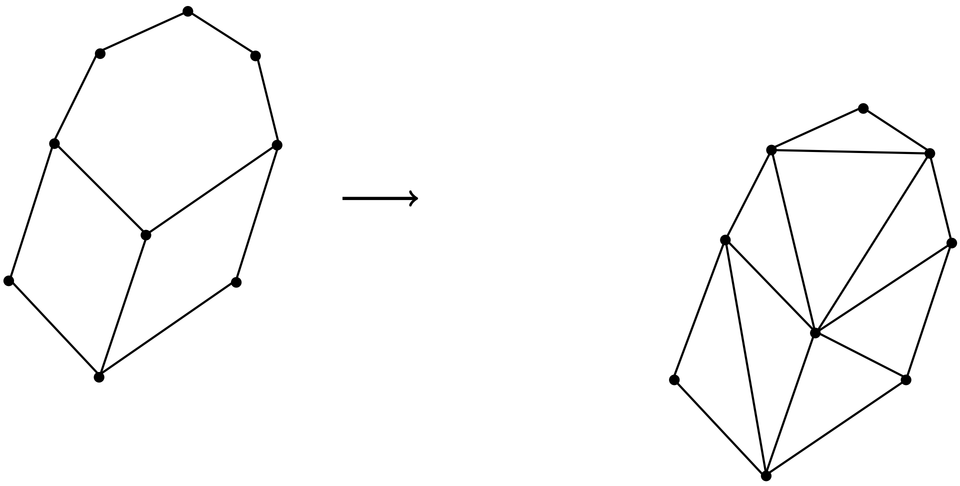
Exercise 9.3. A soccer ball is made of 12 pentagonal and 20 hexagonal pieces. How many sections of seams are there (i.e., edges)? In this question you will be asked to answer this question in two different ways. For part (a) you should *not* use the polyhedron formula; for part (b) you should.

(a) By counting the number of seams on each face, and the number of faces that each seam touches, determine the answer. (You are using combinatorics to solve it, rather than appealing to Theorem 9.1.)

(b) After determining the number of "vertices" on a soccer ball (perhaps by using a counting approach similar to the one from part (a)), apply the polyhedron formula to find the answer.

Exercise 9.6. In this exercise, you will investigate Augustin-Louis Cauchy's proof of Theorem 9.1, which was the first one to use the technique of proving the result for any planar graph, not just the ones which correspond to a polyhedron.

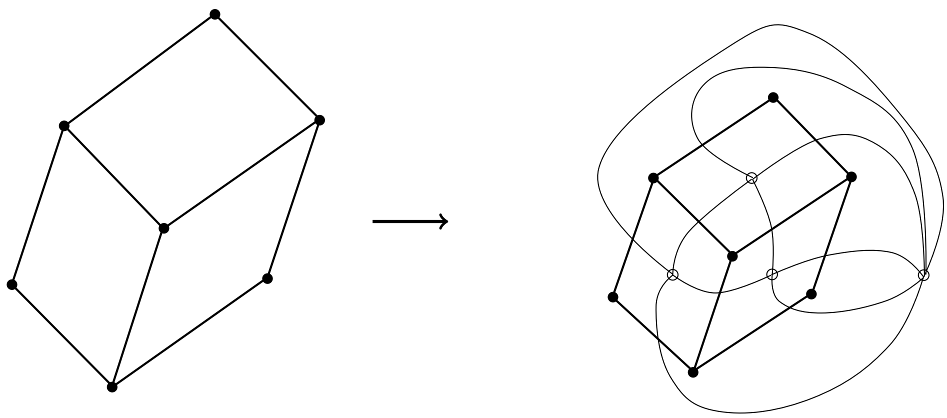
Given any planar graph, add edges to it so that every interior face is a triangle; there are typically many ways to do this. For example:



Argue that the value of $V - E + F$ before these edges were added is the same as after these edges were added. Suppose that after adding these edges, the new graph has N of these interior triangles (in the above example, $N = 8$). Argue that there is a way to remove certain vertices and edges so that the remaining picture has precisely $N - 1$ triangles, and the value of $V - E + F$ has not changed. Then, argue that you can repeat your approach $N - 2$ more times giving you just one triangle. Show that at this point we have $V - E + F = 2$, and consequently the original graph must have had $V - E + F = 2$, as well.

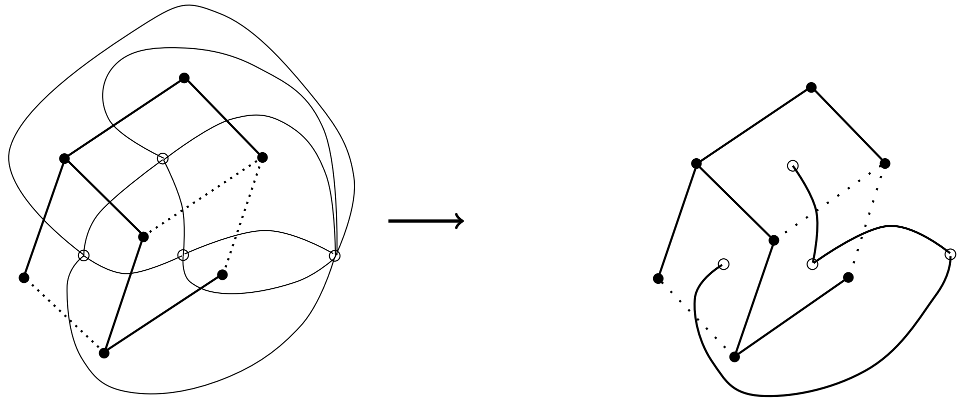
Exercise 9.8. In this exercise, you will prove Theorem 9.1 by proving the result holds for any planar graph. You will use a proof by induction, by inducting on the number of faces in a planar graph. In your induction step, consider an edge between two faces, and see what happens if that edge is removed.

Exercise 9.9. In this exercise, you will prove Theorem 9.1 using what is called the *dual* of a graph. Given a polyhedron with E edges, V vertices, and F faces, we first draw it as a planar graph. Then, we find the dual of this graph, which is another graph obtained by placing a vertex in the middle of each face (including the outer face), and if two faces share an edge, we connect their corresponding vertices with an edge (which might result in more than one edge between a pair of vertices). For example:



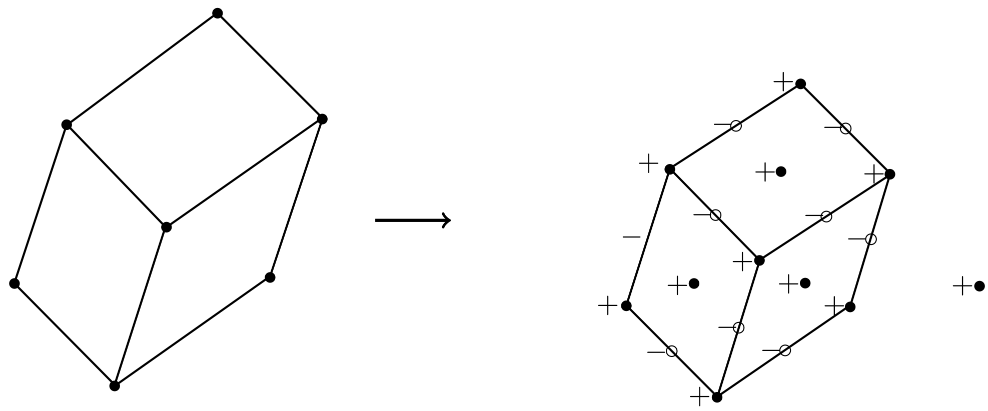
Next, look at any *spanning tree* in the original graph,[1] and the edges in the dual graph that this spanning tree does not cross.

[1] That is, you want to find edges in the graph that are connected to each other (you don't want one component to be disconnected from another component) and if the graph has n vertices, your spanning tree will have exactly $n - 1$ edges. This will mean that the tree does not have any "cycles." You can look up more examples online if any of these ideas are new to you.



Explain why this procedure produces a spanning tree of the dual graph. Then, explain why $E = (V-1) + (F-1)$, and how this completes the proof.

Exercise 9.10. In this exercise, you will prove Theorem 9.1 using another form of the magical discharging method we used in this chapter's proof of the theorem. Draw your polyhedron as a planar graph, and when you do so, ensure that no edge is drawn vertically. Then, add positive and negative charges to the graph in the same way we did in the chapter: A positive charge to each vertex and face, and a negative charge to each edge. For example:



Now, we discharge. Because none of the edges are vertical, each edge has one vertex on its left and one on its right. Discharge each edge's negative charge to its right vertex. Likewise, each face (except the outer face) has a right-most vertex; discharge these face charges to the right-most vertex. Explain how this proves the result.

Theorem 9.1 (Euler's Polyhedron Formula)**.** *For any convex polyhedron with V vertices, E edges, and F faces,*

$$V - E + F = 2.$$

More generally, this formula holds for any connected planar graph, where F counts the number of faces (regions) including the unbounded outer face.